Journal of Scientific Research and Studies Vol. 9(1), pp. 8-16, February, 2022
ISSN 2375-8791

http://www.modernrespub.org/jsrs/index.htm

Full Length Research Paper

A shorter solution to the Clay millennium problem about regularity of the Navier-Stokes equations

Konstantinos E. Kyritsis

Department of Accounting and Finance, University of Ioannina, Psathaki Preveza, 48100, Greece. E-mail: ckiritsi@uoi.gr, C_kyrisis@yahoo.com

Accepted 15 February, 2022

The Clay millennium problem regarding the Navier-Stokes equations is one of the seven famous difficult and significant mathematical problems. Although it is known that the set of Navier-Stokes equations has a unique smooth local time solution under the assumptions of the millennium problem, it is not known whether this solution can always be extended for all times smoothly, which is called the regularity (no blow-up) of the Navier-Stokes equations in 3 dimensions. Of course, the natural outcome would be that the regularity also holds for 3 dimensions since we know that it holds in 2 dimensions. Compared to the older solution proposed by Kyritsis (2021a) for the non-periodic setting without external forcing, this paper solves it also for the case with the periodic setting without external forcing. The strategy is based again in discovering new momentum density invariants derived from the well-known Helmholtz-Kelvin-Stokes theorem of the velocity circulation.

Key words: Incompressible flows, regularity, Navier-Stokes equations, 4th Clay millennium problem.

Mathematical Subject Classification: 76A02

INTRODUCTION

My first attempt to solve the millennium problem about the regularity of the Navier-Stokes equations problem was during the spring 2013 (uploaded at that time see references Kyritsis, 2013) and eventually in Kyritsis (2018) and Kyritsis (2021a). The solution has also been published as a chapter in a book by Kyritsis (2021c) and also as a whole book devoted to it by Kyritsis (2021d). The latter book also contains the solution of the 3rd Clay Millennium problem in computational complexity, since the author has solved 2 of the 7 millennium problems (Kyritsis, 2021b). The author has also solved the 3rd Millennium problem P vs NP in computational complexity with 3 different successive solutions each one simpler than the previous (see references Kyritsis (2019) and Kyritsis (2021b) and Kyritsis (2021d)). In the paper by Kyritsis (2017b), there is also an alternative 3rd solution to the 4th Clay Millennium problem based on the additional hypothesis of the conservation of particles. The other solution has no additional hypothesis other than those of the formulation of the problem by the Clay Mathematical Institute.

The main objective of the current paper is to solve it also (regularity, no blow-up of the Navier-Stokes equations) for **the periodic case too** with initial data as in the standard formulation of the 4th Clay Millennium Problem for the homogeneous case (no external forcing; See last paragraph 4). It is proved that in the periodic case too, without external forcing, not only there is no Blow-up in finite time but not even at the time T=+∞.

An attempt has been made to keep the length of this paper as short as possible to encourage reading it and to make the solution as easy to understand as possible.

The main core of the solution is the Paragraph 3. In this paragraph 3 is discussed what it is that probably we do not understand with the Navier-Stokes equations, and the well-known Helmholtz-Kelvin-Stokes theorem of velocity circulation is extended to new momentum density invariants of the flow. Based on these new invariants, we are able to prove in theorem 3.4 that the vorticity cannot blow up (thus regularity). The paragraph 2 is devoted to reviewing the standard formulation of the 4th Clay Millennium problem, while the 4th paragraph simply applies trivially the results of the paragraph 3 **to solve the 4th Clay Millennium problem as well for the periodic homogeneous case (no external forcing).**

THE STANDARD FORMULATION OF THE 4TH MILLENNIUM CONJECTURE ABOUT THE NAVIER-STOKES EQUATIONS AND SOME CRITERIA OF REGULARITY.

In this paragraph, we highlight the basic parts of the standard formulation of the 4th Clay millennium problem as in Fefferman C.L. 2006.

The **Navier-Stokes** equations are given by (by R we denote the field of the real numbers, v>0 is the density normalized viscosity coefficient):

$$\frac{\partial}{\partial t}u_i + \sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + v\Delta u_i \quad (x\in R^3,\ t>=0,\ n=3) \qquad (1)$$

$$div u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0 \quad (x\in R^3,\ t>=0,\ n=3) \qquad (2)$$

(a) with initial conditions u(x,0)=u⁰(x) x∈R³

(b) and u⁰ (x) ∈ C^∞ divergence-free vector field on R³ (3)

If v=0, then we are taking about the **Euler** equations and **inviscid** case.

(c) $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ **is the Laplacian operator. The Euler equations are Equations 1, 2 and 3 when v=0.**

It is reminded to the reader that in the equations of Navier-Stokes, as in Equation 1, the density ρ is constant, and it is custom to normalized to 1 and omit it.

(d) For physically meaningful solutions, we want to make sure that u⁰(x) does not grow large as |x|→∞. This is set by defining u⁰(x) and f(x,t) and is called in this paper as Schwartz initial conditions to satisfy

$\left|\partial_x^{a} u^0(x)\right| \le C_{a,K}(1+|x|)^{-K}$ on **R³** for any *α* and K (4)

(Schwartz used such functions to define the space of Schwartz distributions)

Remark 2.1: It is important to realize that smooth Schwartz initial velocities after Equation 4 will give that the initial vorticity ω_0 =curl(u^0) in its supremum norm, which is bounded over all 3-space.

$\left|\partial_x^{a}\partial_t^{m} f \quad (x,t)\right| \le C_{a,m,K}(1+|x|+t)^{-K}$ on **R³**× $[\mathbf{0},+\infty)$ for any *α*,*m*,K (5)

We accept as physical meaningful solutions only if it satisfies

p, u ∈$C^\infty(R^3\times[0,\infty))$ (6)

and

$\int_{\Re^3} |u(x,t)|^2\, dx < C$ for all t>=0 (**Bounded or finite energy**) (7)

Remark 2.2: It is important to realize that smooth external force (densities) with the Schwartz property as in Equation 5 have not only a rule for upper bounded spatial partial derivatives, but also the same rule for time upper bounded partial derivatives.

Remark 2.3: We must stress here that imposing smoothness of the coordinate functions of velocities and external forces of the initial t=0 data and later time t data in the Cartesian coordinates plus and Schwartz condition as in Equation 5 is not equivalent with imposing similar such smoothness of the coordinate functions and conditions in the cylindrical or spherical coordinates. We will give in paragraph 4, remark 4.5 an example of a strange blowup, where at any time t>0, the coordinates of the velocities are smooth and bounded in all space as functions in the polar coordinates and still the vorticity has infinite singularity at zero.

Alternatively, to rule out problems at infinity, we may look for spatially periodic solutions of Equations 1, 2 and 3. Thus we assume that $u^0(x)$ and f(x,t) satisfy

$u^0(x+e_j)= u^0(x)$, $f(x+e_j,t)= f(x,t)$,
$p(x+e_j,0)=p(x,0)$, for 1<=j<=3 (8)

(e_j is the jth unit vector in R^3)

In place of Equations 4 and 5, we assume that $u^0(x)$ is smooth and that

$\left|\partial_x^{a}\partial_t^{m} f \quad (x,t)\right| \le C_{a,m,K}(1+t)^{-K}$ on **R³**× $[\mathbf{0},+\infty)$ for any *α*,*m*,K (9)

We then accept a solution of Equations 1, 2 and 3 as physically reasonable if it satisfies

$u(x+e_j ,t)= u(x, t)$, $p(x+e_j, t)=p(x,t)$, on **R³**× $[\mathbf{0},+\infty)$ for 1<=j<=3 (10)

and p, u ∈$C^\infty(R^3\times[0,\infty))$ (11)

In the next paragraphs, we may also write u_0 instead of u^0 for the initial data velocity.

We denote Euclidean balls by $B(a,r):=\{x\in R^3: ||x-a||\leq r\}$, where ||x|| is the Euclidean norm.

The 4 sub-problems or conjectures of the millennium problem are the next:

(Conjecture A) Existence and smoothness of Navier-Stokes solution on R^3.
Take v>0 and n=3. Let $u_0(x)$ be any smooth, divergent-free vector field satisfying Equation 4. Take f(x,t) to be identically zero. Then there exist smooth functions p(x,t) and u(x,t) on R^3x[0,+∞) that satisfy Equations 1, 2, 3, 6 and 7.

(Conjecture B) Existence and smoothness of Navier-Stokes solution on R^3/Z^3.
Take v>0 and n=3. Let $u_0(x)$ be any smooth, divergent-free vector field satisfying Equation 8; we take f(x,t) to be identically zero. Then there exist smooth functions p(x,t), u(x,t) on R^3x[0,+∞) that satisfy Equations 1, 2, 3, 10 and 11.

(Conjecture C) Breakdown of Navier-Stokes solution on R^3
Take v>0 and n=3. Then there exist a smooth, divergent-free vector field $u_0(x)$ on R^3 and a smooth f(x,t) on R^3x[0,+∞) satisfying Equations 4 and 5 for which there exist no smooth solution (p(x,t) and u(x,t)) of Equations 1, 2, 3, 6 and 7 on R^3x[0,+∞).

(Conjecture D) Breakdown of Navier-Stokes solution on R^3/Z^3
Take v>0 and n=3. Then there exist a smooth, divergent-free vector field $u_0(x)$ on R^3 and a smooth f(x,t) on R^3x[0,+∞) satisfying Equations 8 and 9 for which there exist no smooth solution (p(x,t) and u(x,t)) of Equations 1, 2, 3, 10 and 11 on R^3x[0,+∞).

Remark 2.4: It is stated in the same formal formulation of the Clay millennium problem by C. L. Fefferman (2006) (see page 2, line 5 from below) that the conjecture (A) has been proved to hold locally. "..if the time internal [0,∞) is replaced by a small time interval [0,T), with T depending on the initial data....". In other words, there is ∞>T>0, such that there exists a unique and smooth solution $u(x,t)\in C^\infty(R^3\times[0,T))$. See also A.J. Majda-A.L. Bertozzi, Theorem 3.4 pp. 104. In this paper, as it is standard almost everywhere, the term smooth refers to the space C^∞

In the next the $|| ||_m$ is the corresponding Sobolev spaces norm and we denote it by V^m ={u in $H^m(R^n)$ and divu=0}, where $H^m(R^n)$ are the Sobolev spaces with the L^2 norm. We must mention that in A.J. Majda-A.L. Bertozzi, Theorem 3.4 pp. 104, Local in Time existence of Solutions to the Euler and Navier-Stokes equations, it is proved that indeed if the initial velocities belong to V^m m>=[3/2]+2, there exist unique smooth solutions locally in time [0,t]. Here, in the formulation of the millennium problem, the hypotheses of smooth with Schwartz condition initial velocities satisfies this condition; therefore, we have the existence and uniqueness of smooth solution locally in time, both in the non-periodic and the periodic setting without external forcing (homogeneous case).

The existence and uniqueness of a smooth solution locally in time is stated in the formulation by C.L. Fefferman for the homogeneous cases and Conjectures A and B. When a smooth Schwartz condition external force is added (inhomogeneous case), it is natural to expect that also there should exist a local in time unique sooth solution. However, I did not find this to be stated in the A.J. Majda-A.L. Bertozzi**, so I will avoid assuming it.**

We state here also two, very well-known criteria of no blow-up and regularity.

In this theorem, the $|| ||_m$ is the corresponding Sobolev spaces norm and we denote it by V^m ={u in $H^m(R^n)$ and divu=0}, where $H^m(R^n)$ are the Sobolev spaces with the L^2 norm.

Theorem 2.1: Velocities Sobolev norm sufficient condition of regularity. *Given an initial condition $u_0\in V^m$ m>=[3/2]+2=3.5 e.g. m=4, then for any viscosity v>=0, there exists a maximal time T* (possibly infinite) of existence of a unique smooth solution $u\in C([0,T^*];V^m)\cap C^1([0,T^*];V^{m-2})$ to the Euler or Navier-Stokes equation. Moreover, if T*<+∞, then necessarily $\lim_{t\to T^*}||u(.\,,t)||_m=+\infty$.*

Proof: See A.J. Majda-A.L. Bertozzi, Corollary 3.2 pp. 112). QED

Remark 2.5: Obviously, this proposition covers the periodic case too.

Theorem 2.2: Supremum of vorticity sufficient condition of regularity
Let the initial velocity $u_0\in V^m$ m>=[3/2]+2, e.g. m=4, so that there exists a classical solution $u\in C^1([0,T];C^2\cap V^m)$ to the 3D Euler or Navier-Stokes equations. Then:

(i) If for any T>0 there is $M_1>0$ such that the vorticity ω=curl(u) satisfies $\int_0^T |\omega(.,\tau)|_{L^\infty}\, d\tau\leq M_1$
Then the solution u exists globally in time, $u\in C^1([0,+\infty];C^2\cap V^m)$

(ii) If the maximal time T^* of the existence of the solution $u\in C^1([0,T];C^2\cap V^m)$ is finite, then necessarily the vorticity accumulates so rapidly that

$$lim_{t\to T^*}\int_0^T |\omega(.,\tau)|_{L^\infty}\, d\tau=+\infty \quad (12)$$

Proof: See A.J. Majda-A.L. Bertozzi, Theorem 3.6 pp. 115, L^∞ vorticity control of regularity.

QED.

Remark 2.6: Obviously, this proposition covers the periodic case too.

SOME NEW MOMENTUM DENSITY INVARIANTS OF THE NAVIER-STOKES EQUATIONS AFTER THE HELMHOLTZ-KELVIN-STOKES THEOREM.

In the current solution of the millennium problem, we utilize versions of the fundamental theorem of the calculus (Stokes theorem, etc.) These versions of the fundamental theorem of the calculus (Stokes theorem, etc.) lead to an extension of the law of momentum conservation of 3D fluid parts to a law of 1D line density (rotatory) momentum conservation (Theorem 3.1) and law of 2D surface density (rotatory) momentum conservation (Theorem 3.2). These laws are very valuable for infinite divisible fluids, thus valuable as the existence of finite atoms in the atomics structured fluids. Without these extra laws of momentum density conservation, we would have no hope to solve the millennium problem. As T. Tao had remarked, only an integral of 3D energy conservation and an integral of 3D momentum conservation is not adequate to derive that momentum point densities ρ•u, or energy point densities $(1/2)\rho\bullet u^2$ will not blow up.

Besides the forgotten conservation law of finite particles, which unfortunately we cannot utilize in the case of infinite divisible fluids to solve the millennium problem, there are **two more forgotten laws of conservation or invariants**. The first of them is obvious that during the flow, the physical measuring unit's dimensions (dimensional analysis) of the involved physical quantities (mass density, velocity, vorticity, momentum, energy, force point density, pressure, etc.) are conserved. It is not very wise to eliminate the physical magnitudes interpretation and their dimensional analysis when trying to solve the millennium problem because the dimensional analysis is a very simple and powerful interlink of the involved quantities and leads with the physical interpretation to a transcendental shortcut to symbolic calculations. **By eliminating the dimensional analysis, we lose part of the map to reach our goal.**

The 2nd forgotten conservation law or invariant is related to the viscosity (friction). Since we do know that at each point (pointwise), the viscosity is only subtracting kinetic energy, with an irreversible way, and converting it to thermal energy (negative energy point density), and this is preserved in the flow (it can never convert thermal energy to macroscopic kinetic energy), knowing fully well that its sign does not also change and is a flow invariant, so the integrated 1D or 2D work density is always of the same sign (negative) and as sign, an invariant of the flow. The **conservation or invariance of the sign of work density by the viscosity (friction)** is summarized in Lemma 3.1 below.

Finally, we must not understate the elementary fact that the force densities F_p due to the pressures p, $F_p = -\nabla p$ are conservative, irrotational vector field, and they do not contribute to the increase or decrease of the rotatory momentum and vorticity of the fluid during the flow. Because of this, we get that the conserved 1D and 2D densities of momentum in Theorems 3.1 and 3.2 are only of the rotatory type.

Anyone who has spent time to try to prove existence of Blow up or regularity in the various physical quantities of the fluid like velocity, vorticity, acceleration, force density, momentum, angular momentum, energy, etc. will observe that in the arguments, the regularity and uniform in time boundedness propagates easily from derivatives to lower order of differentiation, while the blowup arguments propagate easily from the magnitudes to their derivatives. The converses are hard in proving. This is due to the usual properties of the calculus derivatives and integrals. The hard part of the proofs must utilize forms of the fundamental theorem of the calculus like Stokes theorem, divergence theorem, etc.

Based on the above remarks about what is not very well understood with Navier-Stokes equations, I decided that **elementary geometric calculus should be the appropriate context to solve the millennium problem**, and this I did indeed.

Here in the next, we apply the idea that the most valuable equations governing the flow of the fluid are not literally the Navier-Stokes equations but the invariants or semi-invariant properties of the flow, derived from the abstract multi-dimensional fundamental theorems of calculus, in the forms of divergence theorems, Stokes theorems, Greens theorems, etc. Actually, this is the mechanism of wedge-products and abstract algebra of differential forms, which is beyond classical partial differential equations. We do not utilize definitions and symbolism of wedge-products and differential forms in this paper so as to keep it elementary and easy to read. The main discovery of this paragraph is the **Helmholtz-Kelvin-Stokes theorem 3.3 in the case of viscous flows and the resulting general no-blow-up theorem 3.4 for the viscous flows without external forcing.** A blow-up, when it occurs, will occur at least as blow-up of the vorticity, or of ρ•ω. If we discover average value invariants of the flow with physical units dimensions ρ•ω, which in the limit can give also the point value of the ρ•ω, **and that are invariants independent from the size of averaging,** it is reasonable that we can deduce conclusions if the point densities can blow-up or not.

Theorem 3.1: The *Helmholtz-Kelvin-Stokes theorem* in the case of inviscid Euler equations flows without external force or homogeneous case. (A 1D line density of rotatory momentum, conservation law).

Let the initial data in R^3 be that they guarantee the existence of a unique smooth solution to the Euler equation in a local time interval [0,T]. Then at any time t ∈ [0,T], the circulation Γ(c) of the velocities on a closed smooth loop is equal to the flux of the vorticity on smooth surface S with boundary the loop c, and is constant and preserved as both loop and surface flow with the fluid. In symbols (ρ=1 is the density of the incompressible fluid),

$$\Gamma_{c(t)} = \rho \oint_{c=\partial S} u dl = \rho \iint_S \omega \bullet ds \qquad (13)$$

Proof: See Majda, A.J-Bertozzi, A. L. 2002, Proposition 1.11 and Corollary 1.3, in page 23. The proof is carried actually by integrating the Euler equations on a loop c and utilizing that the integral of the pressure forces (densities) defined as –∇p are zero as it is a conservative (irrotational) field of force (densities). Then by applying also the Stokes theorem that makes the circulation of the velocity on a loop equal to the flux of the vorticity on a smooth surface with boundary the loop (see e.g. Wikipedia Stokes theorem https://en.wikipedia.org/wiki/Stokes%27_theorem) claim is obtained. QED.

We may notice that this circulation and surface vorticity flux has physical measuring units [ρ]*[ω]*[s]^2=[m]*[s]^(-3)*[t]^(-1)[s]^2=[m]*[s]^(-1)*[t]^(-1) =[moment_of_inertia]*[ω]*[s]^(-3) thus angular momentum point density. While the ρ*ω has physical measuring units dimensions [ρ]*[ω]=[m]*[s]^(-3]*[t]^(-1) =[moment_of_inertia]*[ω]*[s]^(-2) thus 2nd spatial derivative of rotational momentum of point density.

In a blow-up when it occurs, it will occur at least as blow-up of the vorticity or of ρ•ω. If we discover bounded average value invariants of the flow with physical units dimensions ρ•ω, which in the limit can give also the point value of the ρ•ω, and that are invariants and bounded independent from the size of averaging, it is reasonable that we can deduce conclusions if the point densities can blow-up or not.

Here we convert the surface vorticity flux invariant of Helmholtz-Kelvin-Stokes to one with 3D integration, which will be more convenient in the arguments as the volumes are preserved by incompressible flows and, most importantly, the integration is 3-dimensional that can be utilized to define average values of the vorticity (flux) on 3D finite particles.

We will prove at first a lemma about the 3D volume integral of Theorem 3.2 and convergence of average values of vorticity based on this 3D integral to point values to vorticity.

We define an average value for the volume 3D integral of vorticity flux.

Definition 3.1: *We define as average value on ball in the vorticity ω, denoted by $\overline{\omega}_B$, the unique constant value of the vorticity on the interior of the ball that would give the same 3D flux of vorticity on the ball, $\rho\int_0^{\pi} \iint_S \overline{\omega} \bullet ds d\theta$ = $\rho\int_0^{\pi} \iint_S \omega \bullet ds d\theta$. The integration on the surface S for the flux of the vorticity is on parallel circular discs of the ball. This average value $\overline{\omega}$ of the vorticity is of course the*

$$||\overline{\omega}_B|| = |\frac{\rho\int_0^{\pi} \iint_S \omega \bullet ds d\theta}{|B|}| \qquad (14)$$

*and its direction is that of the vertical axis of the ball B, where |B|=(4/3)*π*r³ is the volume of the ball B, of radius r, and $||\overline{\omega}_B||$ is the Euclidean norm of the vector. A more detailed symbolism of the average vorticity is the $\overline{\omega}(x_t, t)_{B(r,t)}$*

*The numerator of this average value of vorticity has also the interpretation of **rotational momentum average axial density** on the ball B and relative to the axis a. A reason for this is that the physical dimensions of measuring units of this magnitude is that of rotational momentum line density. This is because the rotational momentum point density has physical dimensions [moment_of_inertia]*[ω]*[s]^(-3)=[m][s]^(-1)[t]^(-1), where [m] is mass, [s] is distance, [t] is time, and this magnitude has physical units dimensions, ([ρ][ω][s]^3)=([m][s]^(-1)[t]^(-1))[s]^(1), thus rotational momentum point density is integrated on 1-d line axial density. And the full quotient therefore has physical unit dimensions [m][s]^(-3)[t]^(-1))=[ρ][ω].*

Lemma 3.1: *Let a ball B of radius r and center x have the average vorticity $\overline{\omega}_B$ in it as in Definition 4.1 such that its axis a that defines the average vorticity is also the axis of the point vorticity ω_x at the center x of the ball. By taking the limit of shrinking the ball to its center x, (r->0), the average vorticity $\overline{\omega}_B$ converges to the point vorticity ω_x. In symbols $\lim_{r\to 0} \overline{\omega}_B = \omega_x$. If the axis a of the ball to estimate the average vorticity is not the axis of the point vorticity, then the limit of the average vorticity will be equal to the projection component* ω_a(x,t) *of the point vorticity* ω(x,t) *on the axis a.*

Proof: We simply apply an appropriate 3-dimensional version with iterated integrals of the 1-dimensional fundamental theorem of the calculus. QED.

Remark 3.1: Such a limit of 3D body to a point is the same as the limit as that from the Newton equation of force F=mγ, thus deriving the Navier-Stokes equations.

Since the flow of a fluid under the Euler or Navier-Stokes equations with or without smooth Schwartz external force is a smooth and continuous mapping F, then such a limit will be conserved to still be a valid limit during the flow. In other words, F($\lim_{B\to 0} \overline{\omega}_B$)= $\lim_{F(B)\to 0} F(\overline{\omega}_B)$ and B->0 implies F_t(B)->0. We define of course in an obvious appropriate way the average vorticity $F_t(\overline{\omega}_B)$ as in Definition 3.1, for the flow-image of a ball B after time t. Simply the disc surfaces will no longer be flat and the loop no longer perfect circle. However, the integrals in the definition will be the same. Constancy of the average vorticity on such surfaces will only be up to its Euclidean norm and vertical angle to the surface. We must notice though that although a relation F($\lim_{B\to 0} \overline{\omega}_B$)= $\lim_{B\to 0} F(\overline{\omega}_B)$ would hold, the value of this limit **will not be** the vorticity $\omega_{F(x)}$ at the flowed point! Unfortunately, the Lemma 3.2 holds not on arbitrary 3D shapes and arbitrary integration parametrization on it, but only when we start with standard 3D shapes like a sphere, a cylinder, a cube, etc. and the normal parametrization on them. The reason is that we need to take into account in a normal way the average vorticity around a point in an unbiased way, which an arbitrary shape will not give.

Another important conservation point is that the relation of the vorticity ω_x being tangent to an axis a (or general curve) is conserved during inviscid Euler flows. It is the conservation of vorticity lines (See Majda, A. J. –Bertozzi, A. L. 2002, Proposition 1.9 in page 21). Therefore for inviscid (and incompressible) flows, the axis of the initial point vorticity ω(0), which is also the axis to estimate the average vorticity on the ball B, will still be after the flow and at time t, tangent to the point vorticity ω(t). However, for general viscous flows, this will not be so. Notice that such limits of average values would not work for the circulation of the velocity on a loop, as in the application of the iterated 1-dimensional fundamental theorem of the calculus would require boundaries of the integration.

Lemma 3.2: *Let the Euler or Navier-Stokes equations of incompressible fluids in the non-periodic or periodic setting have smooth initial data and we assume that the initial data in the periodic or non-periodic case are so that the supremum of the* ***vorticity*** *is finite, as denoted by F_ω on all 3-space at time t=0. Let the average vorticity or average rotational momentum density be defined as in Definition 4.1, but with integration parametrization on any smooth 3D shape B of any size, which of course involves a diffeomorphic image of a spherical ball with its spherical coordinates integration parametrization. Then the average vorticity or average rotational momentum density is also upper bounded by the F_ω. In symbols*

$$||\overline{\omega}_B|| = \left|\frac{\int_0^\pi \iint_S \ \omega \bullet ds d\theta}{|B|}\right| \leq \quad F_\omega \qquad (15)$$

Proof: Since ||ω|| <= F_ω **=** **||**(ω/||ω||)||F_ω in the flux-integration, we have for the inner product of ω and the unit area vector n, (ω, n)<= ((ω/||ω||)F_ω ,n)<= F_ω . Thus in the integration, we may factor out the F_ω

$$\left|\frac{\int_0^\pi \iint_S \ \omega \bullet ds d\theta}{|B|}\right| \leq \left|\frac{\int_0^\pi \iint_S \ F_\omega ds d\theta}{|B|}\right| = F_\omega\left|\frac{\int_0^\pi \iint_S \ ds d\theta}{|B|}\right| = F_\omega \left|\frac{|B|}{|B|}\right|$$

= F_ω . QED.

Lemma 3.3: The viscosity sign forgotten invariant.

If we integrate the force point density of the viscosity over a line (1D work density) or surface (2D work density) or a volume (work), its sign will remain the same during the flow.

Proof: Because we do know that pointwise, the viscosity is only subtracting kinetic energy, with an irreversible way, and converting it to thermal energy (negative energy point density), having it being preserved in the flow (it can never convert thermal energy to macroscopic kinetic energy), we deduce that its sign does not change also as a flow invariant, so the integrated 1D or 2D work density is always of the same sign (negative), which is an invariant of the flow. QED.

Theorem 3.2: A 3-dimensional integral version of the Helmholtz-Kelvin-Stokes theorem (A 2D surface density of rotatory momentum and conservation law).

Let the initial data in R^3 be that they guarantee the existence of a unique smooth solution to the Euler equation in a local time interval [0,T]. Then at any time t ∈ [0,T], let a sphere B of radius r be ***considered as a finite particle****, then the azimuthal θ-angle and the θ-integral on a meridian in spherical coordinates of the circulations Γ(c) of the velocities on all closed longitude smooth loops parallel to the equatorial loop is equal to the same θ-integral of the surface flux of the vorticity on smooth flat disc surfaces S with boundary loops c, and both integrals are constant and preserved as both surface and volume integrals during the flow with the fluid as in symbols (ρ=1 is the density of the incompressible fluid).*

$$\rho \int_0^\pi \oint_{c=\partial S} u dl d\theta = \rho \int_0^\pi \iint_S \ \omega \bullet ds d\theta \qquad (16)$$

After Equation 14 || $\overline{\omega}_B|| = \left|\frac{\int_0^\pi \iint_S \ \omega \bullet ds d\theta}{|B|}\right|$ *holds also for t ∈ [0,T]*

$$\overline{|| \ \omega}_{B(0)}|| = ||\overline{\omega}_{B(t)}|| \qquad (17)$$

Proof: We simply take the θ-azimuthal angle θ-integral of both sides of Equation 18 in Theorem 3.1. Both sides are preserved during the flow and so is their θ-integrals too. We notice that the measuring physical units dimensions of the conserved quantity $\rho \int_0^\pi \oint_{c=\partial S} u dl d\theta$ is [mass]*[length]^(-3)*[velocity]*[length]^(2)= [mass]*[length]^(-2)*[velocity], thus integration in 2-dimension surface of momentum 3D-point-density or equivalently momentum 1D density QED

Theorem 3.3: The *<u>Helmholtz-Kelvin-Stokes theorem</u>* in the case of viscous Navier-Stokes equations flows without external force (homogeneous case).

Let initial data in R^3 be that they guarantee the existence of a unique smooth solution to the Navier-Stokes equation with viscosity coefficient v>0, in a local time interval [0,T]. Then at any time t ∈ [0,T], the circulation Γ(c) of the velocities on a closed smooth loop is equal to the flux of the vorticity on smooth surface S with boundary of the loop c and is decreasing as both loop and surface flow with the fluid, as in symbols (ρ=1 is the density of the incompressible fluid).

$$\rho \oint_{c=\partial S} u dl = \rho \iint_S \ \omega \bullet ds \qquad (18)$$

And for t∈[0,T]

$$\oint_{c=\partial S} u(0)dl > \oint_{c=\partial S} u(t)dl \qquad (19)$$

Similarly for the 3D volume integration as in Theorem 3.2

for t ∈ [0,T]

$$\rho \int_0^{\pi} \iint_S \ \omega(0) \bullet dsd\theta > \rho \int_0^{\pi} \iint_S \ \omega(t) \bullet dsd\theta \qquad (20)$$

After Equation 14, $\|\ \overline{\omega}_B \| = |\frac{\int_0^{\pi} \iint_S \ \omega \bullet dsd\theta}{|B|}|$ *holds also for initial finite spherical particles for t ∈ [0,T]*

$$\|\overline{\omega}_{B(0)}\| > \|\overline{\omega}_{B(t)}\| \qquad (21)$$

Proof: Again Equation 18 is nothing else of course but the Stokes theorem. We shall utilize here the next equation (See Majda, A.J-Bertozzi, A. L. 2002, Equation 1.61, in page 23) in the case of viscous incompressible flows under the Navier-Stokes equations

$$\frac{d}{dt}\Gamma_{c(t)} = \frac{d}{dt}\oint_{c(t)} udl = \nu \oint_{c(t)} \Delta udl = -\nu \oint_{c(t)} curl\ \omega dl \quad , \qquad (22)$$

This equation is derived after applying Theorem 3.1 in the loop integral of the circulation at the Navier-Stokes equations instead of at the Euler equations, taking the material-flow derivative outside the integral and eliminating the conservative and irrotational part of the pressure forces as gradient of the pressure. Here, the viscosity is not zero, thus the left hand of the equation is not zero as in the case of Euler equations, where it is conserved. **The right hand side is nothing else than the loop work density of the point density of the force of viscosity at any time t. And as the viscosity always subtracts energy, this right hand side work density is always negative during the flow.** We notice after **Lemma 3.3**, the viscosity force point density keeps constant sign on the trajectory path as orbital component during the flow and relative to the velocity on the trajectory. It is always as an orbital component opposite to the motion and represents the always irreversible energy absorption and linear momentum and angular momentum decrease, which is similar to any rotation of the fluid e.g. with axis on the trajectory path. The viscosity force point density as a component on the loop is always opposite to the rotation, and it never converts thermal energy to add to linear or angular momentum. This is opposite to the motion monotonicity of the viscosity force density, which applies to the Navier-Stokes equations, but also as opposite to rotation monotonicity in the vorticity equation $\frac{D\omega}{Dt} = \omega * \nabla u + v\Delta\omega$ (see Majda, A.J-Bertozzi, A. L. 2002, (Equation 1.33) and (Equation 1.50) in pages 13 and 20).

So if we choose a direction of the loop such that the circulation integral on the right hand side is positive then this will have the same sign during the flow (although different absolute value) and will make the left hand side of the Equation 4.9 always negative during the flow**.** However, this means that from the left-hand side of the equation, the circulation of the velocity on the loop is always decreasing during the flow.

$$\frac{d}{dt}\oint_{c(t)} udl < 0 \qquad \text{for any t in [0,T]} \qquad (23)$$

Thus Equation 19 is proved and Equation 20 is a direct consequence.

To prove Equation 21, we notice that due to incompressibility, the flow is volume preserving, thus |B(x(t))|=|B(x(0)|, and by dividing both sides of Equation 4.7, and after the definition

$\|\ \overline{\omega}_B \| = |\frac{\int_0^{\pi} \iint_S \ \omega \bullet dsd\theta}{|B|}|$ *holds also* for t ∈ [0,T]

$\|\overline{\omega}_{B(0)}\| > \|\overline{\omega}_{B(t)}\|$ (21) QED.

Theorem 3.4: The no blow-up theorem in finite or infinite time in the Euler, Navier-Stokes, periodic or non-periodic and homogeneous cases.

Let the Euler or Navier-Stokes equations of incompressible fluids in the non-periodic or periodic setting (homogeneous case with no external forces) be:

a) Smooth initial data and whatever else hypothesis is necessary so as also to guarantee the ***existence and uniqueness of smooth solutions*** *to the equations locally in time [0, T).*

b) Furthermore, we assume that the initial data in the periodic or non-periodic case are such that the supremum of the vorticity, denoted by F_ω, is finite at t=0 (*in the periodic case, smoothness of the initial velocities is adequate to derive it, while in the non-periodic setting smooth Schwartz initial velocities is adequate to derive it). Then it holds that there cannot exist any finite or infinite time blow-up at the point vorticity during the flow.*

Proof: The proof will be a contradiction. The main idea of the proof is to utilize that in the case of a blow-up, the vorticity will converge to infinite, such that it will become larger than an arbitrary lower bound $M+F_\omega$, $M>0$, $F_\omega>0$ and by approximating it with average flux vorticity of a 3D spherical particle. Besides, tracing it back at the initial conditions where all is bounded by F_ω and utilizing the semi-invariance of the average vorticity that we have proved, we will obtain $F_\omega > M+F_\omega$.

So let us assume that there is a blow up in a finite time or infinite time T^* with the hypotheses of Theorem 4.2. Then from Theorem 2.2 and Equation 12, which is the well-known result of the control of regularity or blow up by the vorticity, we obtain:

$$\lim_{t\to T*}\int_0^T |\omega(.,\tau)|_{L^\infty}\ d\tau = +\infty \quad (12)$$

We conclude that there exist an infinite sequence of points {x_{tn}, n natural number, 0<t_n<T*, $\lim_{n\to\infty} t_n = T^*$ } so that the point vorticity $\omega(x_{tn})$ blows-up, or equivalently $\lim_{n\to\infty}\omega(x_{tn}) = +\infty$. We do not need to assume them on the same trajectory. Therefore, for every positive arbitrary large real number M_0, there is a n_0 such that for all natural numbers n> n_0, it holds that $\omega(x_{tn})>M_0$. We choose $M_0=M_{00}+F_\omega$, for an arbitrary large positive number M_{00}. So

$$\omega(x_{tn})>M_{00}+F_\omega \quad (24)$$

Now we approximate this point vorticity with an average flux vorticity on a 3D particle after Definition 3.1, Theorem 3.2 and Lemma 3.1.

Let a spherical ball particle be B(r, x_{tn}) as shown in Theorem 3.2 with center x_{tn} and radius r>0. After Definition 3.1, Theorem 3.2 and Lemma 3.1, we have that

$$\lim_{r\to 0}\ \overline{\omega}_B = \omega_{x(tn)}\ , \quad \text{With}$$

$$\overline{||\omega_B||} = \left|\frac{\int_0^{2\pi}\iint_S\ \omega\bullet ds d\theta}{|B(r,x(tn))|}\right| \quad (14)$$

Therefore for arbitrary small positive number ε>0, there is radius R, with

$$\overline{\omega}_{B(R)} > \omega_{x(tn)} - \varepsilon$$

$$\text{or } \left|\frac{\int_0^{2\pi}\iint_S\ \omega\bullet ds d\theta}{|B(r,x(tn))|}\right| > \omega_{x(tn)} - \varepsilon \quad (25)$$

Thus after Equation 24,

$$\left|\frac{\int_0^{2\pi}\iint_S\ \omega\bullet ds d\theta}{|B(R,x(tn))|}\right| > M_{00} + F_\omega - \varepsilon \quad (26)$$

Now we trace back on the trajectory of the x_{tn} the parts of Equation 26. At initial time t=0, we use the advantage that as the incompressible flow is volume preserving, the |B(R, x_0,)|= $|B(R, x(tn))|$. We also utilize Theorems 3.2, 3.3, and Equations 17 and 21, which prove that at the initial conditions t=0, this average vorticity is the same or higher than that at t_n .

$$\left|\frac{\int_0^{2\pi}\iint_S\ \omega\bullet ds d\theta}{|B(R,x(0))|}\right| \geq \left|\frac{\int_0^{2\pi}\iint_S\ \omega\bullet ds d\theta}{|B(R,x(tn))|}\right|$$

We conclude that

$$\left|\frac{\int_0^{2\pi}\iint_S\ \omega\bullet ds d\theta}{|B(R,x(0))|}\right| > M_{00} + F_\omega - \varepsilon \quad (27)$$

From Equations 27 and 13 of Lemma 3.2, we conclude that

$$F_\omega > M_{00} + F_\omega - \varepsilon \quad (28)$$

However, M_{00} was chosen in an independent way from ε>0 to be arbitrarily large, while ε>0 can be chosen to be arbitrarily small; therefore, a contradiction. Thus there cannot be any blow-up either in finite or infinite time T*. QED.

Remark 3.2. Infinite initial energy: We must remark that we did not utilize anywhere that the initial energy was finite, only that the vorticity initially has finite supremum. Thus this result of no-blow-up can be with infinite initial energy too. But when applying it to the millennium problem we do have there also that the initial energy is finite.

THE SOLUTION OF THE 4TH CLAY MILLENNIUM PROBLEM ABOUT THE NAVIER-STOKES EQUATIONS IN THE PERIODIC AND NON-PERIODIC HOMOGENEOUS CASE.

We are now in a position to prove the Conjectures (A) and (B), non-periodic and periodic setting, and homogeneous case of the Millennium problem

(Millennium Homogeneous Case A) Existence and smoothness of Navier-Stokes solution on R^3.
Take v>0 and n=3. Let $u_0(x)$ be any smooth, divergent-free vector field satisfying Equation 4. Take f(x,t) to be identically zero. Then there exist smooth functions p(x,t), u(x,t) on R^3x[0,+∞) that satisfy Equations 1, 2, 3, 6 and 7.

Proof: All the hypotheses of the no-blow-up Theorem 4.4 are satisfied. After Remark 2.4, with the current case of the millennium problem there exist indeed a unique smooth solution locally in time [0,t] (after A.J. Majda-A.L. Bertozzi, Theorem 3.4 pp. 104, Local in Time existence of Solutions to the Euler and Navier-Stokes equations). Besides, the Schwartz condition of the initial data guarantees that the supremum of the vorticity is finite at t=0 (see Remark 2.1). Therefore, we conclude by Theorem 4.4 that there cannot be any finite or infinite time blow-up. Thus from **Theorem 2.2 Supremum of vorticity sufficient condition of regularity,** we conclude that this local in time [0,t] solution can be extended in [0,+∞).
QED

(Millennium Homogeneous Case B) Existence and smoothness of Navier-Stokes solution on R^3/Z^3.
Take v>0 and n=3. Let $u_0(x)$ be any smooth, divergent-free vector field satisfying Equation 8; we take f(x,t) to be identically zero. Then there exist smooth functions p(x,t), u(x,t) on R^3x[0,+∞) that satisfy Equations 1, 2, 3, 10 and 11.

Proof: All the hypotheses of the no-blow-up Theorem 4.4 are satisfied. After Remark 2.4, with the current case of the millennium problem there exist indeed a unique smooth solution locally in time [0,t] (after A.J. Majda-A.L. Bertozzi, Theorem 3.4 pp 104, Local in Time existence of Solutions to the Euler and Navier-Stokes equations). Besides, the compactness of the 3D torus of the initial data guarantees that the supremum of the vorticity is finite at t=0. Therefore, we conclude by Theorem 3.4 that there cannot be any finite or infinite time blow-up. Thus from **Theorem 2.2 Supremum of vorticity sufficient condition of regularity** and **remark 2.6 (that the previous theorem covers the periodic setting too),** we conclude that this local in time [0,t] solution can be extended in [0,+∞).

EPILOGUE

In this paper, the regularity (no blow-up) of the Navier-Stokes equations has been proven with the standard assumptions of the formulation of this Millennium problem, and compared to the authors' previous solution in Kyritsis (2021a), **it has also been proved for the periodic** setting (no external forcing).

ACKNOWLEDGEMENTS

I would like to thank professors G. A. Athanasoulis and S. Voutsinas of the National Technical University of Athens for their valuable encouragement and advice on fluid dynamics. I would also like to thank professor Achilles K. Tertikas, Department of Mathematics and Applied Mathematics, University of Crete, for his valuable bibliographic support.

REFERENCES

Beale JT, Kato T, Majda A (1984). Remarks on the breakdown of smooth solutions for the 3-D Euler equations. Commun. Math. Phys. 94:61–66.

Fefferman CL (2006). Existence and smoothness of the Navier-Stokes equation. Millennium Prize Problems, Clay Math. Inst., Cambridge, MA, pp. 57-67.

Kyritsis K (2013). Proof of the existence of global in time smooth solutions of 3-dimensional incompressible inviscid smooth flow-solutions of the Euler equations and viscous smooth flow-solutions of the Navier-Stokes equations (The solution of the Clay millennium problem). An outline of this paper was presented in the 2nd Applied Innovation conference of TEI of Epirus in Arta 16-17 October 2013 (http://innovation-conference-epirus.blogspot.gr/2015/01/2nd-international-conference-on-applied.html) http://libsearch.teiep.gr/Record/cris-123456789-1324

Kyritsis K (2017a). On the solution of the 4th Clay Millennium problem about the Navier-Stokes equations. Equivalence of smooth compact support and smooth Schwartz initial conditions. Proceedings of the 1st international conference on quantitative, social, biomedical and economic issues ICQSBEI 2017 pp 147-172.

Kyritsis K (2017b). On the 4th Clay Millennium problem: Proof of the regularity of the solutions of the Euler and Navier-Stokes equations, based on the conservation of particles. J. Sci. Res. Stud. 4(11):304-317.

Kyritsis K (2018). A solution of the 4th Clay Millennium problem about the Navier-Stokes equations. Preprint 1) http://cris.teiep.gr/jspui/handle/123456789/1628?locale=en 2) https://www.researchgate.net/publication/323389082_A_SOLUTION_OF_THE_4_TH_CLAY_MILLENNIUM_PROBLEM_ABOUT_THE_NAVIER-STOKES_EQUATIONS

Kyritsis K (2019). The solutions of the 3rd and 4th Clay Millennium problems. *https://arxiv.org/abs/1902.07265*

Kyritsis K (2021a). A solution of the 4th Clay Millennium problem about the Navier-Stokes equations. World J. Res. Rev. 13(20):25-40. WJRR.org ISSN 2455-3956

Kyritsis K (2021b). Review of the solutions of the clay millennium problem about P ≠ NP =EXPTIME. World J. Res. Rev. 13(3):21-26 https://www.wjrr.org/vol-13 issue-3

Kyritsis K (2021c). Author of chapter 5 in the book “New Visions in Science and Technology” 6:, 60-69. https://doi.org/10.9734/bpi/nvst/v6/5176F Published: 2021-10-02 Chapter in a book https://stm.bookpi.org/NVST-V6/article/view/4135

Kyritsis K (2021d). Author of a book titled “The Solutions of the 3rd and 4th Millennium Mathematical Problems: The solutions of the Millennium Problem P vs NP in computational complexity and the Millennium problem in fluid dynamics”. Publisher: Lap Lambert Academic Publishing. ISBN 6204725629. https://www.lap-publishing.com/catalog/details//store/gb/book/978-620-4-72562-8/the-solutions-of-the-3rd-and-4th-millennium-mathematical-problems

Majda AJ, Bertozzi AL (2002). Vorticity and incompressible flow. Cambridge University texts in applied mathematics.

Tao T (2013). Localisation and compactness properties of the Navier-Stokes global regularity problem. Analysis & PDE 6:25-107.

Tao T (2014). Finite time blowup for an averaged three-dimensional Navier-Stokes equation. Cornell University Library arxiv.org